Root systems of the complex reflection arrangements ($\mathcal{A}$ (G_{25})and $\mathcal{A}$ (G_{26}))

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Abstract: In this paper we study the root systems of the complex reflection arrangements $\mathcal{A}$ (G_{25})and $\mathcal{A}$ (G_{26}).Where we found coxeter graph and coxtere matrix through the derivation of the angles between the simple roots of the complex reflection groups G25 and G26.

Keywords: Root systems, Arrangement, Coxeter graph.

1. Introduction

We review necessary definitions, notations and theorems of root systems. Bourbaki [6] discussed the relationship between the reflection and the root system, and we discussed the same relationship with more explicate and clear way.In [8] Rabeaa found the lattices of $\mathcal{A}$ (G_{25}) and $\mathcal{A}$ (G_{26})). In this paper we construct the simple root systems of G_{25} and G_{26} depending on the defining polynomial of the complex reflection arrangements of G_{25} and G_{26} also we find the angle between simple root system by using matlab and then we find coxeter matrix, coxeter graph.

Throughout this paper V is a finite dimensional complex vector space. A hyperplane H in V is an affine subspace of dimension ℓ-1.A hyperplane arrangement $\mathcal{A} = (\mathcal{A}, V)$ is a finite set of hyperplanes in V.

The arrangement is called centerless if $\cap_{H\in\mathcal{A}} H = \varnothing$, centered with center $T(\mathcal{A})$ if $T(\mathcal{A}) = \cap_{H\in\mathcal{A}} H = \varnothing$ If $\mathcal{A}$ is centered, then coordinate may be chosen so that hyperplane contains the origin and hence $\mathcal{A}$ is called central. A projective arrangement is a finite set of projective hyperplane in projective space.The product $\mathbb{Q}(\mathcal{A})=\Pi_{H\in\mathcal{A}}\alpha_H$ $(where\ \alpha_H\ is\ a\ linear\ form\ and\ H = \ker \alpha_H)$ is called a defining polynomial of $\mathcal{A}$.we agree that $\mathbb{Q}(\emptyset_\ell) = 1$ is the defining polynomial of $\emptyset_\ell$, where $\emptyset_\ell$ is empty l-arrangement. A reflection on V is a linear transformation on V of finite order with exactly n-1 eigenvalues equal to 1. A reflection group G on V is a finite group generated by reflections on V. The group G is reducible if it is a direct product of two proper reflection subgroups and irreducible otherwise. A finite subgroup G of O (V) is generated by a set of reflections S will be called ***coxeter group***. Take Φ to be a finite set of nonzero vectors in V satisfying: $(R_1)\ \Phi\ spans\ V$, $(R_2)\ s_\alpha\Phi = \Phi\ \forall\ \alpha \in \Phi$. Then defined W to be the group generated by all reflections $s_\alpha, \alpha \in \Phi$ .Call Φ a root system with associated reflection group W.A root system Φ is crystallographic if it satisfies the additional condition $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbf{Z}$ for all $\alpha, \beta \in \Phi$; (see Humphreys[3]).

Definition (2.1)**:** [7, 9]

A finite simple graph Γ= (A, B) is an ordered pair consisting of the set A of vertices and the set B of edges with the following two conditions:

1- A is a finite set
2- B is a collection of 2-element subset of A

The mapping $\Psi : B \to A \times A$ is called an incidence map which maps an edge into a pair of vertices called end-vertices of the edge.

Definition (2.2)**:** [6]

Let V is a finite dimensional complex vector space, and $a_{ij} =< \alpha_i, \alpha_j >$. The matrix (a_{ij}) is called the ***cartan matrix*** of it's root system Φ.

A generalized cartan matrix is a square matrix M= (a_{ij}) such that:

1- For diagonal entries, a_{ij} =0.

2- For non-diagonal entries, $a_{ij} \le 0$.

3- a_{ij} =0 iff a_{ji} =0. Where

$$a_{ij} = 2\frac{<\alpha_i, \alpha_j>}{<\alpha_j, \alpha_j>} \quad , \alpha_i, \alpha_j \text{ are simple root}$$

Definition (2.3)**:** [2]

A ***dynkin diagram*** for Φ which is denoted by D_G is a graph has vertices $\{a_1, \dots, a_n\}$, between any two vertices; we place no edge, one edge, two edges or three edges obtained by the following two methods:

***First method*:** $\eta_{\beta\alpha}\eta_{\alpha}\beta = 4Cos^{2}\theta$

Two vertices connected by the $\eta_{\alpha}\beta\eta_{\beta}\alpha$ edges, an edge is drawn between each non- orthogonal pair of vectors such that:

1- An undirected one edge if they make an angle of $\frac{2\pi}{3}$ radians.

2- A directed two edge if they make an angle of $\frac{3\pi}{4}$ radians.

3- A directed three edge if they make an angle of $\frac{5\pi}{6}$ radians.

***Second method*: Cartan matrix**

The generalized cartan matrices are equivalent to dynking diagrams. A multi-edged diagram corresponds to the non-diagonal cartan matrix elements - a_{12}, - a_{21}, with the number of edges drawn equal to max (-a_{21}, -a_{12}), and an arrow pointing towards non unity elements.

Definition (2.4)**:** [6]

Let V is a finite dimensional complex vector space

(1) A ***coxeter system*** (W,S) is a pair consisting of group W together with a finite subset S⊂W satisfying the following conditions :

1- For s, s'∈S, let m(s, s') be the order of ss' and let A be the set of pairs (s, s') such that m(s, s') is finite.

2- The generating set S and the relations $(ss')^{m(s,s')} = 1$ for (s, s') in A form a presentation of the group W.

(2) Let A be a set of vertices. A ***coxeter matrix*** of type A is a symmetric matrix M=$(m_{ij})_{ij\in A}$whose entries are integers or +∞ and with 1's on the diagonal such that all non-diagonal entries are greater than 1

i.e. m_{ii}=1 for i∈S m_{ij}≥2 for i, j ∈S with i≠ j

(3) A ***coxeter graph*** (Γ, f) is a pair consisting of graph Γ together with a map f from the set of edges of this graph to the set consisting of +∞ and the set of integers ≥3.

A coxeter graph is associated to any coxeter matrix N of type A as follows:

i. The graph Γ has set of vertices A and set of edges the set pairs {i, j} of elements of A.

ii. Vertices i and j are connected iff m_{ij}≥ 3.

iii. The map f associates to the edge {i,j} the corresponding element m_{ij} of M.

Proposition (2.5)**:** [5]

If α_i, α_j ∈ Δ, then there is an integer m_{ij}≥ 1 such that

$(\alpha_i, \alpha_j)/\|\alpha_i\| \ \|\alpha_j\|$ = - cos(π /m_{ij}), m_{ij} is the order $\alpha_i\alpha_j$.

3. Main results

❖ **The complex Reflection Group G_{25}:**

Let **V** is a finite dimensional complex vector space

The defining polynomial of $\mathcal{A}(G_{25})$is $\mathbb{Q}(\mathcal{A}(G_{25}))$ = xyz $\Pi_{0\le i, j\le 2}$ (x +ω^i y + ω^i z) . [7]

The hyperplane arrangement of G_{25} [8]

The hyperplanes of $\mathcal{A}(G_{25})$ where $Hi = ker\, \alpha_{H_i}$, 1≤ i≤12 are:

H_1 : x=0	H_2 : y =0
H_3 : z =0	H_4 : x + y+ z =0
H_5 : x +y +ωz =0	H_6 : x+y+ω^2 z =0
H_7 : x +ωy+z=0	H_8 :x +ωy +ωz=0
H_9 : x +ωy + ω^2 z=0	H_{10} : x +ω^2 y +z =0
H_{11} : x +ω^2 y +ωz =0	H_{12} :x+ω^2 y + ω^2 z =0

Table (1) the hyperplanes of $\mathcal{A}(G_{25})$

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Therefore the set $S_{\Phi}(G_{25})$ is the simple root system for the group G_{25}

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e_1=(1,0, 0)	e_2=(0, 1, 0)
e_3= (0, 0, 1)	e_4= (1, 1, 1)
e_5= (1, 1, -0.5+0.8660i)	e_6= (1, 1, -0.5-0.8660i)
e_7= (1, -0.5+0.8660i ,1)	e_8= (1, -0.5+0.8660i, -0.5+0.8660i)
e_9=(1, -0.5+0.8660i, -0.5-0.8660i)	e_{10}=(1, -0.5-0.8660i, 1)
e_{11}= (1, -0.5-0.8660i , -0.5+0.8660i)	e_{12} = (1, -0.5-0.8660i , -0.5-0.8660i)

Table (2) the simple root of G_{25}

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
F_1		90	90	60	60	60	60	60	60	60	60	60
F_2	90		90	60	60	60	105	105	105	105	105	105
F_3	90	90		60	105	105	60	105	105	60	105	105
F_4	60	60	60		60	60	60	90	90	60	90	90
F_5	60	60	105	60		60	90	60	90	90	60	90
F_6	60	60	105	60	60		90	90	60	90	90	60
F_7	60	105	60	60	90	90		60	60	60	90	90
F_8	60	105	105	90	60	90	60		60	90	60	90
F_9	60	105	105	90	90	60	60	60		90	90	60
F_{10}	60	105	60	60	90	90	60	90	90		60	60
F_{11}	60	105	105	90	60	90	90	60	90	60		60
F_{12}	60	105	105	90	90	60	90	90	60	60	60	

Table (3) the angle between simple root of G_{25}

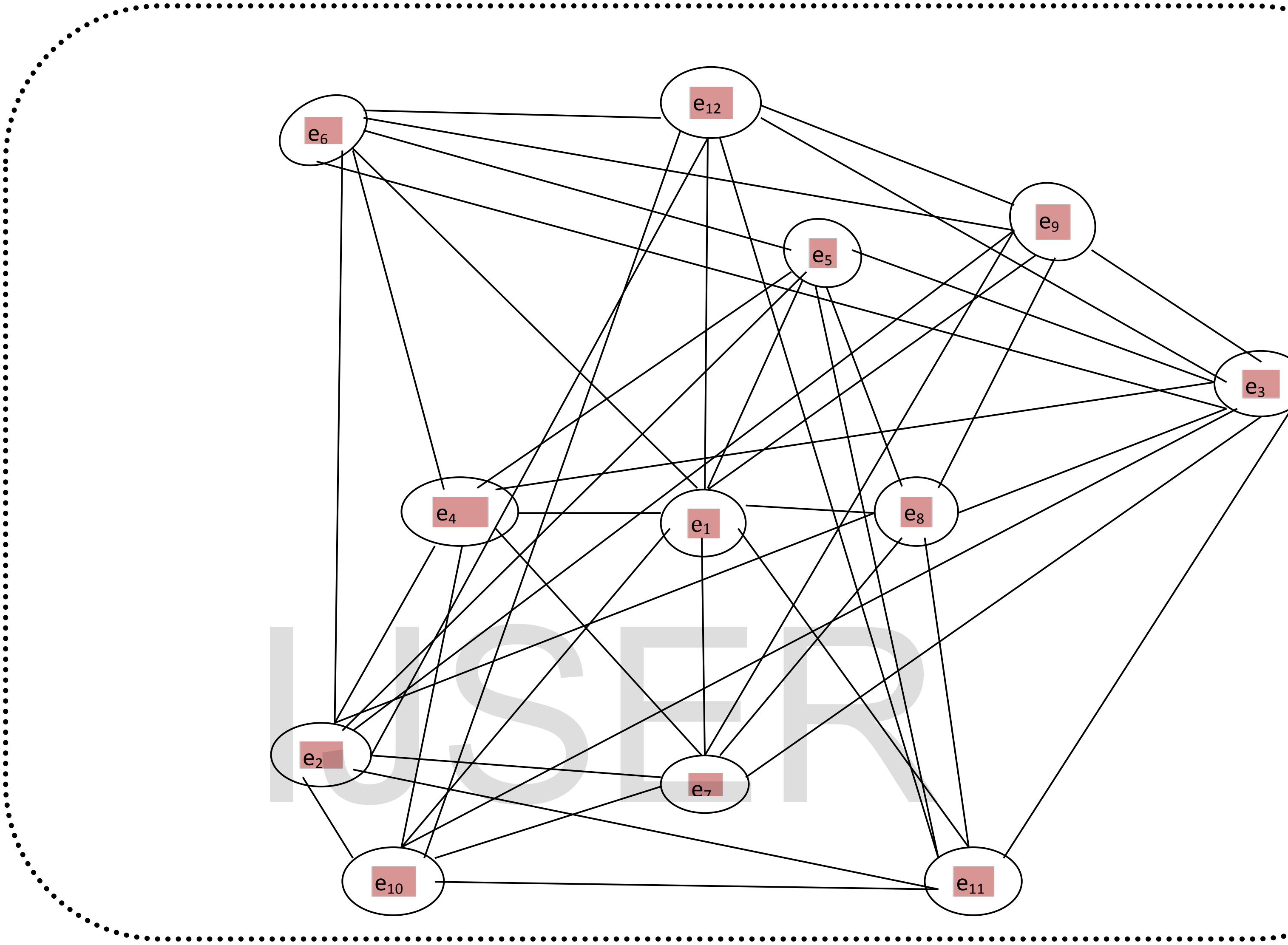
The Coxeter matrix of the complex group G_{25} is

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 1 & 2 & 3 & 3 & 3 & 2.4 & 2.4 & 2.4 & 2.4 & 2.4 & 2.4 \\ 2 & 2 & 1 & 3 & 2.4 & 2.4 & 3 & 2.4 & 2.4 & 3 & 2.4 & 2.4 \\ 3 & 3 & 3 & 1 & 3 & 3 & 3 & 2 & 2 & 3 & 2 & 2 \\ 3 & 3 & 2.4 & 3 & 1 & 3 & 2 & 3 & 2 & 2 & 3 & 2 \\ 3 & 3 & 2.4 & 3 & 3 & 1 & 2 & 2 & 3 & 2 & 2 & 3 \\ 3 & 2.4 & 3 & 3 & 2 & 2 & 1 & 3 & 3 & 3 & 2 & 2 \\ 3 & 2.4 & 2.4 & 2 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\ 3 & 2.4 & 2.4 & 2 & 2 & 3 & 3 & 3 & 1 & 2 & 2 & 3 \\ 3 & 2.4 & 3 & 3 & 2 & 2 & 3 & 2 & 2 & 1 & 3 & 3 \\ 3 & 2.4 & 2.4 & 2 & 3 & 2 & 2 & 3 & 2 & 3 & 1 & 3 \\ 3 & 2.4 & 2.4 & 2 & 2 & 3 & 2 & 2 & 3 & 3 & 3 & 1 \end{bmatrix}$$

3.3 The coxeter graph of the complex group G_{25} is denoted by $C_{G_{25}}$

The cardinality of vertices is 12

Root	Degree of root	Root	Degree of root
e_1	9	e_7	7
e_2	9	e_8	7
e_3	9	e_9	7
e_4	7	e_{10}	7
e_5	7	e_{11}	7
e_6	7	e_{12}	7


e12
e6
e9
e5
e3
e4
e1
e8
e2
e7
e10
e11

- **The complex Reflection Group G_{26}:** The defining polynomial of $\mathcal{A}(G_{26})$ is

$\mathbb{Q}(\mathcal{A}(G_{26}))$ =xyz $\Pi_{0\le i,j\le 2}(x+\omega^i y+\omega^i z)(x^3-y^3)(x^3-z^3)(y^3-z^3)$.[7]

The hyperplane arrangement of G_{26} :The hyperplanes of $\mathcal{A}(G_{26})$ where $Hi = ker\,\alpha_{H_i}$ $1\le i\le 21$ are:

H_1 : x=0	H_2 : y =0
H_3 : z =0	H_4 : x + y+ z =0
H_5 : x +y +ω z =0	H_6 : x+y+ω^2 z =0
H_7 : x +ω y+z=0	H_8 :x +ωy +ω z=0
H_9 : x +ωy + ω^2 z=0	H_{10} : x +ω^2 y +z =0
H_{11} :x +ω^2 y +ω z =0	H_{12} :x+ω^2 y + ω^2 z =0
H_{13} : x – y =0	H_{14} :x – z =0
H_{15} : y – z =0	H_{16} :x – ω y =0
H_{17} :x – ω^2 y =0	H_{18} :x – ω z =0
H_{19} :x – ω^2 z =0	H_{20} :y – ω z =0
H_{21} :y – ω^2 z =0	

Table (4) the hyperplanes of $\mathcal{A}(G_{26})$

Therefore the set $S_{\Phi}(G_{26})$ is the simple root system for the group G_{26} i.e. $S_{\Phi}(G_{26})$ consists of:

e_1=(1, 0, 0)	**e_2=(0, 1, 0)**
e_3=(0, 0, 1)	**e_4=(1, 1, 1)**
e_5=(1, 1, -0.5+0.8660i)	**e_6=(1, 1, -0.5-0.8660i)**
e_7=(1, -0.5+0.8660i,1)	**e_8=(1, -0.5+0.8660i ,-0.5+0.8660i)**
e_9=(1, -0.5+0.8660i ,-0.5-0.8660i)	**e_{10}=(1, -0.5-0.8660i, 1)**
e_{11}=(1, -0.5-0.8660i, -0.5+0.8660i)	**e_{12}=(1, -0.5-0.8660i, -0.5-0.8660i)**
e_{13}=(1, -1,0)	**e_{14}=(1, 0,-1)**
e_{15}=(0, 1,-1)	**e_{16}=(1,0.5-0.8660i, 0)**
e_{17}=(1,0.5+0.86601, 0)	**e_{18}=(1,0,0.5-0.8660i)**
e_{19}=(1, 0, 0.5+0.8660i)	**e_{20}=(0, 1, 0.5-0.8660i)**
e_{21}=(0, 1, 0.5+0.8660i)	

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	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}	F_{21}
F_1		90	90	60	60	60	60	60	60	60	60	60	45	45	90	45	60	45	45	90	90
F_2	90		90	60	60	60	105	105	105	105	105	105	135	90	45	75	30	90	90	45	45
F_3	90	90		60	105	105	60	105	105	60	105	105	90	135	135	90	90	75	75	75	75
F_4	60	60	60		60	60	60	90	90	60	90	90	90	90	90	45	30	45	45	45	45
F_5	60	60	105	60		60	90	60	90	90	60	90	90	45	45	45	30	90	45	90	45
F_6	60	60	105	60	60		90	90	60	90	90	60	90	45	45	45	30	45	90	45	90
F_7	60	105	60	60	90	90		60	60	60	90	90	45	90	135	90	90	45	45	90	90
F_8	60	105	105	90	60	90	60		60	90	60	90	45	45	90	90	90	90	45	135	90
F_9	60	105	105	90	90	60	60	60		90	90	60	45	45	90	90	90	45	90	90	135
F_{10}	60	105	60	60	90	90	60	90	90		60	60	45	90	135	45	90	45	45	90	90
F_{11}	60	105	105	90	60	90	90	60	90	60		60	45	45	90	45	90	90	45	135	90
F_{12}	60	105	105	90	90	60	90	90	60	60	60		45	45	90	45	90	45	90	90	135
F_{13}	45	135	90	90	90	90	45	45	45	45	45	45		60	120	75	105	60	60	120	120
F_{14}	45	90	135	90	45	45	90	45	45	90	45	45	60		60	60	60	75	75	105	105
F_{15}	90	45	135	90	45	45	135	90	90	135	90	90	120	60		75	60	105	105	75	75
F_{16}	45	75	90	45	45	45	90	90	90	45	45	45	75	60	75		45	60	60	75	75
F_{17}	60	30	90	30	30	30	90	90	90	90	90	90	105	60	60	45		60	60	60	60
F_{18}	45	90	75	45	90	45	45	90	45	45	90	45	60	75	105	60	60		75	60	105
F_{19}	45	90	75	45	45	90	45	45	90	45	45	90	60	75	105	60	60	75		105	60
F_{20}	90	45	75	45	90	45	90	135	90	90	135	90	120	105	75	75	60	60	105		75
F_{21}	90	45	75	45	45	90	90	90	135	90	90	135	120	105	75	75	60	105	60	75	

Table (6) the angle between simple root of G_{26}

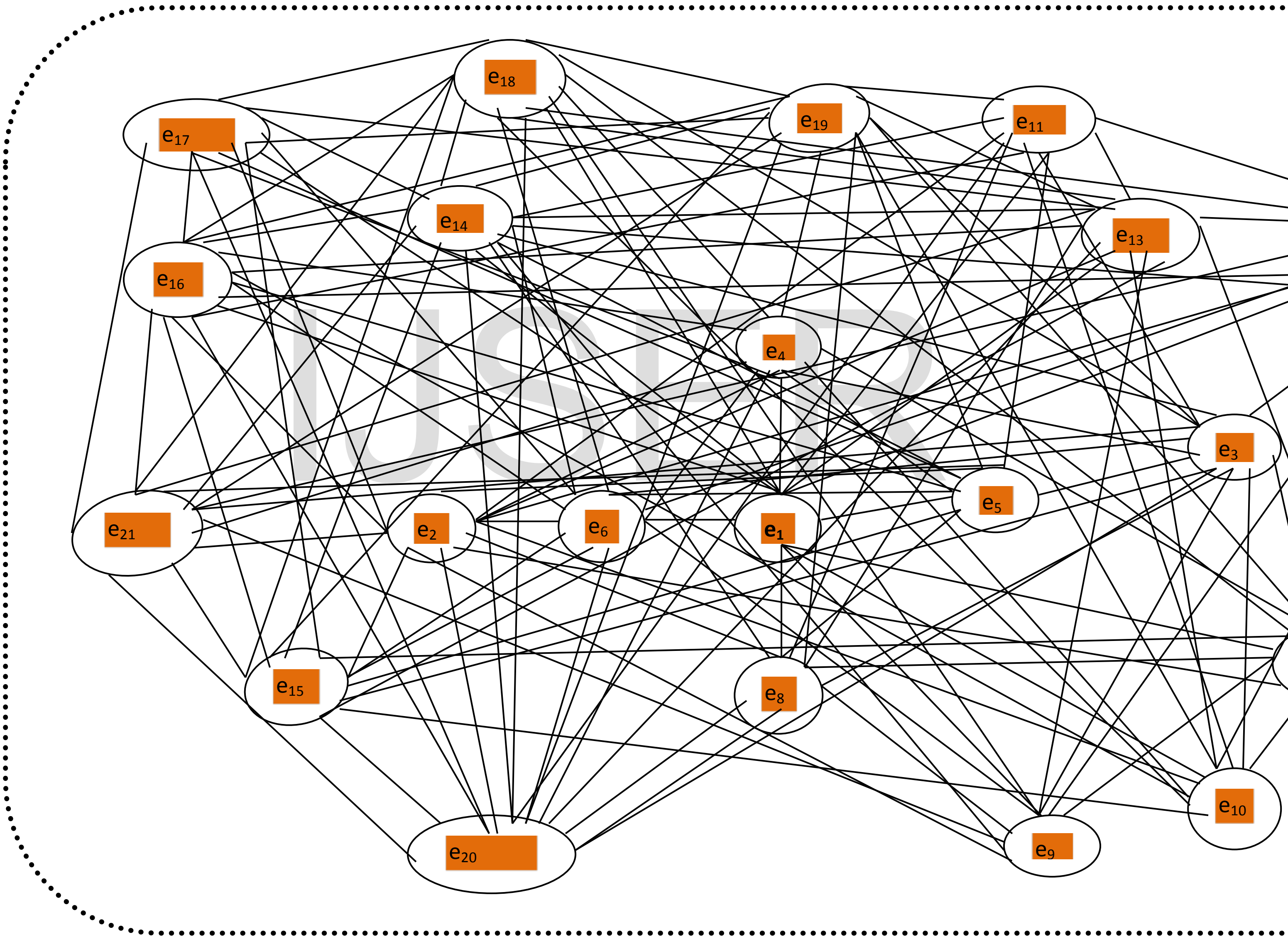
3.5 The Coxeter matrix of the complex group G_{26} is

$$\begin{bmatrix}
1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 2 & 4 & 3 & 4 & 4 & 2 & 2 \\
2 & 1 & 2 & 3 & 3 & 3 & 2.4 & 2.4 & 2.4 & 2.4 & 2.4 & 2.4 & 4 & 2 & 4 & 2.4 & 6 & 2 & 2 & 4 & 4 \\
2 & 2 & 1 & 3 & 2.4 & 2.4 & 3 & 2.4 & 2.4 & 3 & 2.4 & 2.4 & 2 & 4 & 4 & 2 & 2 & 2.4 & 2.4 & 2.4 & 2.4 \\
3 & 3 & 3 & 1 & 3 & 3 & 3 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 4 & 6 & 4 & 4 & 4 & 4 \\
3 & 3 & 2.4 & 3 & 1 & 3 & 2 & 3 & 2 & 2 & 3 & 2 & 2 & 4 & 4 & 4 & 6 & 2 & 4 & 2 & 4 \\
3 & 3 & 2.4 & 3 & 3 & 1 & 2 & 2 & 3 & 2 & 2 & 3 & 2 & 4 & 4 & 4 & 6 & 4 & 2 & 4 & 2 \\
3 & 2.4 & 3 & 3 & 2 & 2 & 1 & 3 & 3 & 3 & 2 & 2 & 4 & 2 & 4 & 2 & 2 & 4 & 4 & 2 & 2 \\
3 & 2.4 & 2.4 & 2 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 2 & 4 & 4 & 2 & 2 & 2 & 2 & 4 & 4 & 2 \\
3 & 2.4 & 2.4 & 2 & 2 & 3 & 3 & 3 & 1 & 2 & 2 & 3 & 4 & 4 & 2 & 2 & 2 & 4 & 2 & 2 & 4 \\
3 & 2.4 & 3 & 3 & 2 & 2 & 3 & 2 & 2 & 1 & 3 & 3 & 4 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 2 \\
3 & 2.4 & 2.4 & 2 & 2 & 3 & 2 & 2 & 3 & 3 & 1 & 3 & 4 & 4 & 2 & 4 & 2 & 2 & 4 & 4 & 2 \\
3 & 2.4 & 2.4 & 2 & 2 & 3 & 2 & 2 & 3 & 3 & 3 & 1 & 4 & 4 & 2 & 4 & 2 & 4 & 2 & 2 & 4 \\
4 & 4 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & 3 & 3 & 2.4 & 2.4 & 3 & 3 & 3 & 3 \\
4 & 2 & 4 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 4 & 4 & 3 & 1 & 3 & 3 & 3 & 2.4 & 2.4 & 2.4 & 2.4 \\
2 & 4 & 4 & 2 & 4 & 4 & 4 & 2 & 2 & 4 & 2 & 2 & 3 & 3 & 1 & 2.4 & 3 & 2.4 & 2.4 & 2.4 & 2.4 \\
4 & 2.4 & 2 & 4 & 4 & 4 & 2 & 2 & 2 & 4 & 4 & 4 & 2.4 & 3 & 2.4 & 1 & 4 & 3 & 3 & 2.4 & 2.4 \\
3 & 6 & 2 & 6 & 6 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2.4 & 3 & 3 & 4 & 1 & 3 & 3 & 3 & 3 \\
4 & 2 & 2.4 & 4 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 4 & 3 & 2.4 & 2.4 & 3 & 3 & 1 & 2.4 & 3 & 2.4 \\
4 & 2 & 2.4 & 4 & 4 & 2 & 4 & 4 & 2 & 4 & 4 & 2 & 3 & 2.4 & 2.4 & 3 & 3 & 2.4 & 1 & 2.4 & 3 \\
2 & 4 & 2.4 & 4 & 2 & 4 & 2 & 4 & 2 & 2 & 4 & 2 & 3 & 2.4 & 2.4 & 2.4 & 3 & 3 & 2.4 & 1 & 2.4 \\
2 & 4 & 2.4 & 4 & 4 & 2 & 2 & 2 & 4 & 2 & 2 & 4 & 3 & 2.4 & 2.4 & 2.4 & 3 & 2.4 & 3 & 2.4 & 1
\end{bmatrix}$$

3.6 The coxeter graph of the complex group G_{26} is denoted by $C_{G_{26}}$ the

Cardinality of vertices is 21

Root	Degree of root	Root	Degree of root	Root	Degree of root
e_1	15	e_8	11	e_{15}	14
e_2	15	e_9	11	e_{16}	16
e_3	15	e_{10}	12	e_{17}	13
e_4	13	e_{11}	12	e_{18}	16
e_5	13	e_{12}	12	e_{19}	17
e_6	13	e_{13}	16	e_{20}	14
e_7	11	e_{14}	16	e_{21}	14


e18
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e3
e21
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e20
e9
e10

References

[1] A Cohen, Finit complex reflection groups, Ann.scient, (1976).

[2] B Hall, Lie group, Lie Algebras, and Representations, An Elementary Introduction, springer, (2003).

[3] J Humphreys, Reflection groups and Coxeter group, Cambridge University press, (1990).

[4] J Martinet, perfect Lattices in Euclidean Spaces , springer – Verlag, Berlin (2003).

[5] L Grove; C Benson, Finite Reflection Groups, (2rd ed) , springer-verlag , New York, (1984).

[6] N Bourbaki, Lie groups and Lie algebras, chapters 4-6, Elements of Mathematics, springer – Verlag, Brlin, (2002).

[7] P Orlik ; H Terao, Arrangements of hyperplanes, Grundlehren Math, Springer, Verlag, Berline, 300 (1992).

[8] R Al-Aleyawee, The Hypersolvable and Free Arrangements. Ph.D.Thesis ,Coll.Science , Univ . AL-Mustansiriyah, (2005).

[9] Xu Junming, Theory and Application of Graphs , Network Theory and Applications, Kluwer Academic pulbishers,10 (2003).